On gradient bounds for the heat kernel on the Heisenberg group

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Abstract

It is known that the couple formed by the two dimensional Brownian motion and its Lévy area leads to the heat kernel on the Heisenberg group, which is one of the simplest sub-Riemannian space. The associated diffusion operator is hypoelliptic but not elliptic, which makes difficult the derivation of functional inequalities for the heat kernel. However, Driver and Melcher and more recently H.-Q. Li have obtained useful gradient bounds for the heat kernel on the Heisenberg group. We provide in this paper simple proofs of these bounds, and explore their consequences in terms of functional inequalities, including Cheeger and Bobkov type isoperimetric inequalities for the heat kernel.

Keywords: Heat kernel ; Heisenberg group ; functional inequalities ; hypoelliptic diffusions
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Contents

1 Introduction

Gradient bounds had proved to be a very efficient tool for the control of the rate of convergence to equilibrium, quantitative estimates on the regularization properties of heat kernels, functional inequalities such as Poincaré, logarithmic Sobolev, Gaussian isoperimetric inequalities for heat kernel measures. The reader may take a look for instance at [3, 38, 39, 28, 2] and references therein. When dealing with the simplest examples, such as linear parabolic evolution equations (or heat kernels), those gradient bounds often rely on the control of the intrinsic Ricci curvature associated to the generator of the heat kernel. Those methods basically require some form of ellipticity of the generator.

The elliptic case

Let $\mathcal{M}$ be a complete connected Riemannian manifold of dimension n and let L be the associated Laplace-Beltrami operator, written in a local system of coordinates as

$$L(f)(x) = \sum_{i,j=1}^{n} a_{i,j}(x)\partial^2_{x_i x_j} f(x).$$

The coefficients $x \mapsto a_{i,j}(x)$ are smooth and the symmetric matrix $(a_{i,j}(x))_{1\leq i,j\leq n}$ is positive definite for every x. The "length of the gradient" $|\nabla f|$ of a smooth $f : \mathcal{M} \to \mathbb{R}$ is given by

$$\Gamma(f,f) = |\nabla f|^2 = \frac{1}{2}(L(f^2) - 2fLf) = \sum_{i,j=1}^{n} a_{i,j}(x)\partial_{x_i} f \partial_{x_j} f.$$

Let $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$ be the heat semigroup generated by L. For every smooth $f : \mathcal{M} \to \mathbb{R}$, the function $(t,x) \mapsto P_t(f)(x)$ is the solution of the heat equation associated to L

$$\partial_t P_t(f)(x) = LP_t(f)(x) \quad \text{and} \quad P_0(f)(x) = f(x).$$

For every real number $\rho \in \mathbb{R}$, the following three propositions are equivalent (see [3, 28, 37]).

1. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \mathrm{Ricci}(\nabla f, \nabla f) \geq \rho\, |\nabla f|^2$
2. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \forall t \geq 0,\ |\nabla P_t f|^2 \leq e^{-2\rho t}\, P_t(|\nabla f|^2)$
3. $\forall f \in \mathcal{C}_c^\infty(\mathcal{M}),\ \forall t \geq 0,\ |\nabla P_t f| \leq e^{-\rho t}\, P_t(|\nabla f|)$

This is the case for some $\rho \in \mathbb{R}$ when $\mathcal{M}$ is compact. This is also the case with $\rho = 0$ when $\mathcal{M}$ is $\mathbb{R}^n$ equipped with the usual metric since Ricci $\equiv 0$. In this last example, L is the usual Laplace operator Δ and the explicit formula for the heat kernel gives $\nabla P_t f = P_t \nabla f$ for the usual gradient ∇ and thus $|\nabla P_t f| \leq P_t\, |\nabla f|$. Back to the general case, and following [3], the gradient bounds 2. or 3. above are equivalent to their infinitesimal version at time $t = 0$, which reads

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f)$$

where

$$\Gamma_2(f,f) = \frac{1}{2}(L\Gamma(f,f) - 2\Gamma(f,Lf)) = |\nabla\nabla f|^2 + \mathrm{Ric}(\nabla f, \nabla f).$$

The bound $\Gamma_2 \geq \rho\Gamma$ had proved to be a very efficient criterion for the derivation of gradient bounds for more general Markov processes, including for instance processes generated by an operator L with a first order linear part (i.e. with a potential).

In the equivalence above, one may add several other inequalities, including local Poincaré inequalities, local logarithmic Sobolev inequalities, and local Bobkov isoperimetric inequalities, and their respective reverse forms, with a specific constant involving $e^{-\rho t}$, see [3, 28]. Here the term local means that they concern the probability measure $P_t(\cdot)(x)$ for any fixed t and x. One may also replace in these inequalities $e^{-\rho t}$ by any function $c(t)$ continuous and differentiable at $t = 0$ with $c(0) = 1$ and $c'(0) = -\rho$. In the present paper, we will focus on the Heisenberg group, a non elliptic situation where these equivalences do not hold, but where some gradient bounds are still available and provide local inequalities of various types.

The Heisenberg group

In recent years, some focus had been set on some degenerate situations, where the methods used for the elliptic case do not apply. One of the simplest example of such a situation is the Heisenberg group (see section 2 for the group structure). Namely, we consider on $\mathbb{H} = \mathbb{R}^3$ the vector fields

$$X = \partial_x - \frac{y}{2}\partial_z \quad \text{and} \quad Y = \partial_y + \frac{x}{2}\partial_z$$

and the operator

$$L = X^2 + Y^2 = \partial_x^2 + \partial_y^2 + \frac{1}{4}(x^2 + y^2)\partial_z^2 + x\partial_{y,z}^2 - y\partial_{x,z}^2. \tag{1}$$

This operator is self-adjoint for the Lebesgue measure on $\mathbb{R}^3$. The matrix of second order derivatives associated to L is degenerate and thus L is not elliptic. If $[U, V] = UV - VU$ stands for the commutator of U and V, then

$$Z := [X, Y] = \partial_z \quad \text{and} \quad [X, Z] = [Y, Z] = 0.$$

In particular, L is hypoelliptic in the Hörmander sense (the Lie algebra described by $\{X, Y, Z\}$ is the Lie algebra of the Heisenberg group, see section 2). As a consequence, the heat semigroup $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$ obtained by solving the heat equation associated to L admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^3$. It is remarkable that the Markov process associated to this semigroup is the couple formed by a Brownian motion on $\mathbb{R}^2$ and its Lévy area, and for every fixed $t > 0$ and $\mathbf{x} \in \mathbb{H}$, the probability distribution $P_t(\cdot)(\mathbf{x})$ is a sort of Gaussian on $\mathbb{H}$. We refer to [7] and [36] for such probabilistic aspects. For this operator L we have also

$$\Gamma(f, f) = X(f)^2 + Y(f)^2 \tag{2}$$

and

$$\Gamma_2(f, f) = X^2(f)^2 + Y^2(f)^2 + \frac{1}{2}(XY + YX)(f)^2 + \frac{1}{2}(Zf)^2 + 2(XZ(f)Y(f) - YZ(f)X(f)).$$

The presence of $YZ(f)$ and $XZ(f)$ in the Γ_2 expression forbids the existence of a constant $\rho \in \mathbb{R}$ such that $\Gamma_2 \geq \rho\Gamma$ as functional quadratic forms. Therefore the methods used in the elliptic case to prove gradient bounds could not work. In other words, the Ricci tensor is everywhere $-\infty$. In fact, a closer inspection of the Ricci tensor of the elliptic operator $X^2 + Y^2 + \epsilon Z^2$ when ϵ goes to 0 shows that this operator has everywhere a Ricci tensor which is

$$\begin{pmatrix} -\frac{1}{2\epsilon} & 0 & 0 \\ 0 & -\frac{1}{2\epsilon} & 0 \\ 0 & 0 & \frac{1}{2\epsilon^2} \end{pmatrix}.$$

In the limit, one may consider that the lower bound of the Ricci tensor for L is everywhere $-\infty$. Despite this singularity, B. Driver and T. Melcher proved in [16] the existence of a finite positive constant C_2 such that

$$\forall f \in \mathcal{P}^\infty(\mathbb{H}),\ \forall t \geq 0,\ \ |\nabla P_t f|^2 \leq C_2 P_t(|\nabla f|^2). \tag{3}$$

where $\mathcal{P}^\infty(\mathbb{H})$ is the class of smooth function from $\mathbb{H}$ to $\mathbb{R}$ with all partial derivatives of polynomial growth. Here C_2 is the best constant, i.e. the smallest possible. As in the elliptic case, the gradient bound (3) implies a Poincaré inequality for P_t, since

$$P_t(f^2) - (P_t f)^2 = 2\int_0^t P_s(|\nabla P_{t-s} f|^2)\,ds \leq 2tC_2 P_t(|\nabla f|^2). \tag{4}$$

The gradient bound (3) gives also a reverse Poincaré inequality for P_t, since

$$P_t(f^2) - (P_t f)^2 = 2\int_0^t P_s(|\nabla P_{t-s} f|^2)\,ds \geq \frac{2t}{C_2}\,|(\nabla P_t f)|^2\,. \tag{5}$$

¿From the point of view of regularization, (5) is the most important, while (4) is more concerned with estimates on the heat kernel and concentration properties. More recently, H.-Q. Li showed in [29] that there exists a finite positive constant C_1 such that

$$\forall f \in \mathcal{P}^\infty(\mathbb{H}),\ \forall t \geq 0,\ \ |\nabla P_t f| \leq C_1 P_t(|\nabla f|). \tag{6}$$

It is shown in [16] that $C_1 \geq \sqrt{2}$ and $C_2 \geq 2$. The Jensen or Cauchy-Schwarz inequality for P_t gives $C_2 \leq C_1^2$, however, the exact values of C_1 and C_2 are not known to the authors knowledge. The gradient bound (6) is far more useful than (3), and has for instance many consequences in terms of functional inequalities for P_t, including Poincaré inequalities, Gross logarithmic Sobolev inequalities, Cheeger type inequalities, and Bobkov type inequalities, as presented in section 6. As we shall see later, (6) is much harder to obtain than (3).

More generally, one may consider for any $p \geq 1$ and $t \geq 0$ the best constant $C_p(t)$ in $[0,\infty]$ (i.e. the smallest possible, possibly infinite) such that

$$\forall f \in \mathcal{P}^\infty(\mathbb{H}),\ \ |\nabla P_t f|^p \leq C_p(t) P_t(|\nabla f|^p).$$

It is immediate that $C_p(0) = 1$. According to [16] and [29], for every $p \geq 1$ and $t > 0$, the quantity $C_p(t)$ belongs to $(1,\infty)$ and does not depend on t. In particular, C_p is discontinuous at $t = 0$, and this reflects the fact that the Γ_2 curvature of L is $-\infty$.

The aim of this paper is mainly to provide simpler proofs of the gradient bounds (3) and (6). We also give in section 6 a collection of consequences of (6) in terms of functional inequalities for the heat kernel of the Heisenberg group. Section 2 gathers some elementary properties of the Heisenberg group used elsewhere. Section 3 provides a direct simple proof of the reverse Poincaré inequality (5) without using (3) or (6). Sections 4 and 5 provide elementary proofs of (3) and (6) respectively.

2 Elementary properties of the Heisenberg group

We summarize in this section the main properties of the Heisenberg group that we use in the present paper. For more details on the geometric aspects, we refer to [20, 34, 25]. The link with the Brownian motion and its Lévy area is considered for instance in [7] and [36]. From now on, we shall use the notations

$$\langle f\rangle \quad \text{and} \quad \langle f, g\rangle = \langle fg\rangle$$

to denote the integral of a function f with respect to the Lebesgue measure in $\mathbb{R}^3$ and the scalar product of two functions f, g in $\mathrm{L}^2(\mathbb{R}^3,\mathbb{R})$. The Heisenberg group $\mathbb{H}$ is the set of matrices

$$M(x,y,z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

equipped with the following non-commutative product

$$M(x,y,z)M(x',y',z') = M(x+x', y+y', z+z'+xy').$$

The inverse of $M(x,y,z)$ is $M(-x,-y,xy-z)$. It is often more convenient to work with the Lie algebra of the group. For that, we define

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and we consider

$$N(x,y,z) = \exp(xX + yY + zZ),$$

which gives

$$N(x,y,z) = M\left(x, y, z + \frac{xy}{2}\right).$$

We shall therefore identify a point $\mathbf{x} = (x,y,z)$ in $\mathbb{R}^3$ with the matrix $N(x,y,z)$ and endow $\mathbb{R}^3$ with this group structure that we denote $\mathbf{x} \bullet \mathbf{y}$ which is

$$(x,y,z) \bullet (x',y',z') = (x+x', y+y', z+z'+\frac{1}{2}(xy'-yx')).$$

The left invariant vector fields which are given by

$$X(f) = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} \bullet (\epsilon,0,0)) - f(\mathbf{x})}{\epsilon} = (\partial_x - \frac{y}{2}\partial_z)(f),$$

$$Y(f) = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} \bullet (0,\epsilon,0)) - f(\mathbf{x})}{\epsilon} = (\partial_y + \frac{x}{2}\partial_z)(f),$$

$$Z(f) = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} \bullet (0,0,\epsilon)) - f(\mathbf{x})}{\epsilon} = \partial_z(f),$$

while the right invariant vector fields are given by

$$\hat{X}(f) = \lim_{\epsilon \to 0} \frac{f((\epsilon,0,0) \bullet \mathbf{x}) - f(\mathbf{x})}{\epsilon} = (\partial_x + \frac{y}{2}\partial_z)(f),$$

$$\hat{Y}(f) = \lim_{\epsilon \to 0} \frac{f((0,\epsilon,0) \bullet \mathbf{x}) - f(\mathbf{x})}{\epsilon} = (\partial_y - \frac{x}{2}\partial_z)(f).$$

And $\hat{Z} = Z$ since for the points on the z axis, left and right multiplications coincide. The Lie algebra structure is described by the identities $[X,Y] = Z$ and $[X,Z] = [Y,Z] = 0$. In what follows, we are mainly interested in the operator $L = X^2 + Y^2$, and the associated heat semigroup $(P_t)_{t \geq 0} = (e^{tL})_{t \geq 0}$. We shall make a strong use of symmetries in what follows. They are described by the Lie algebra of the vector fields which commute with L. This Lie algebra is 4-dimensional and is generated by the vector fields $\hat{X}$, $\hat{Y}$, Z, and $\Theta = x\partial_y - y\partial_x$. The first ones, which correspond to the right action, commute with X, Y, Z (as it is the case on any Lie group), the last one reflects the rotational invariance of L. There is also another vector field which plays an important role : the dilation operator D, described by

$$D = \frac{1}{2}(x\partial_x + y\partial_y) + z\partial_z.$$

This operator D satisfies

$$[L, D] = L. \tag{7}$$

Let $(T_t)_{t \geq 0} = (e^{tD})_{t \geq 0}$ be the group of dilations generated by D, that is

$$T_t f(x, y, z) = f(e^{t/2}x, e^{t/2}y, e^t z).$$

¿From the commutation relations, one deduces

$$P_t T_s = T_s P_{e^s t}, \tag{8}$$

and

$$P_t D = D P_t + t P_t L. \tag{9}$$

Since P_t commutes with left translations, if $l_{\mathbf{x}}(f)(\mathbf{y}) = f(\mathbf{x}\mathbf{y})$, then

$$P_t(f)(\mathbf{x}) = l_x P_t(f)(0) = P_t(l_x f)(0).$$

Here we have just formalized the fact that $(P_t)_{t \geq 0}$ is a heat semigroup on a group. Moreover, since 0 is a fixed point of the dilation group, one has from equation (8)

$$P_t(f)(0) = P_1(T_{\log t} f)(0).$$

This explains why $P_1(f)(0)$ gives the whole $(P_t f)_{t \geq 0}$. It is well known that

$$P_1(f)(0) = \int_{\mathbb{R}^3} f(\mathbf{y}) h(\mathbf{y}) d\mathbf{y},$$

where $d\mathbf{y}$ is the Lebesgue measure on $\mathbb{R}^3$ and the function h has the following Fourier representation in the z variable

$$h(x, y, z) = \frac{1}{8\pi^2} \int_{-\infty}^{+\infty} e^{i\lambda z} \exp\left(-\frac{r^2}{4}\lambda \coth \lambda\right) \frac{\lambda}{\sinh \lambda} d\lambda. \tag{10}$$

where $r^2 = x^2 + y^2$ if $\mathbf{x} = (x, y, z)$. This formula appeared independently in the works of Gaveau and Lévy. It is not easy to deduce from this formula any good estimates on h, and it is not even easy to see that h is positive. Nevertheless, there are quite precise bounds on this function and its derivatives, see for instance [18, 19] and [29, 30], which may be expressed in terms of the Carnot-Carathéodory distance. The Carnot-Carathéodory distance may be defined as

$$d(\mathbf{x}, \mathbf{y}) = \sup_{\{f \text{ such that } \Gamma(f,f) \leq 1\}} f(x) - f(y)$$

where Γ is as in (2). Here, the explicit form is not easy to write, but we shall only need to express the distance from 0 to $\mathbf{x}$. In order to do that, it is better to describe the constant speed geodesics starting from 0. First, straight lines in the (x, y) plane passing through $(0, 0)$ are geodesics, and the other ones are helices, whose orthogonal projection on the horizontal plane $\{z = 0\}$ is a circle containing the origin (see [7]). If a point $\mathbf{x}(s)$ is moving at constant speed (for the Carnot-Carathéodory distance on the geodesic, its horizontal projection $\mathbf{p}(s)$ moves with unit speed (for the Euclidean distance) on this circle.

Moreover, the height $z(t)$ of the point $\mathbf{x}(s)$ is the surface between the segment $(0, \mathbf{p}(s))$ and the circle, which comes from the fact that for any curve in $\mathbb{R}^3$ whose tangent vector is a linear combination of X and Y, one has

$$dz = \frac{1}{2}(x dy - y dx),$$

that is the area spanned by the point $\mathbf{p}(s)$ in the plane. Note that all the geodesics end on the z axis, which is therefore the cut-locus of the point 0. Those geodesics may be parameterized as follows, using a complex notation for the horizontal projection $\mathbf{p}(s)$

$$\mathbf{p}(s) = u\left(1 - \exp\left(\frac{is}{|u|}\right)\right) \quad \text{and} \quad z(s) = \frac{|u|^2}{2}\left(\frac{s}{|u|} - \sin\left(\frac{s}{|u|}\right)\right). \tag{11}$$

Here, u is the center of the circle which is the horizontal projection of the geodesic, and s is distance from 0 ($s \in (0, 2\pi |u|)$). When $|u|$ goes to infinity, we recover the straight lines.

If we call $d(\mathbf{x})$ the Carnot-Carathéodory distance from 0 to $\mathbf{x}$, it is easy to see from that if $\mathbf{x} = (x, y, z)$, and $s = d(\mathbf{x})$, then

$$d\left(\frac{x}{s}, \frac{y}{s}, \frac{z}{s^2}\right) = 1.$$

This corresponds to a change of u into $\frac{u}{s}$. Now, the unit ball for the Heisenberg metric is between two balls for the usual Riemannian metric of $\mathbb{R}^3$, and therefore one concludes easily that the ratio

$$\frac{(x^2+y^2)^2+z^2}{d^4(x,y,z)}$$

is bounded above and below. Although, if $R = ((x^2+y^2)^2+z^2)^{1/4}$, the function $\Gamma(R, R)$ is bounded above but not below.

Let h be the heat kernel density at time 1 and at the origin, given by (10). The main properties of h used in the present paper are the following. Here for real valued functions a and b, we use the notation $a \simeq b$ when the ratio a/b is bounded above and below by some positive constants. First,

$$h(\mathbf{x}) \simeq \frac{\exp(-\frac{d^2(\mathbf{x})}{4})}{\sqrt{1 + \|\mathbf{x}\|\, d(\mathbf{x})}}, \tag{12}$$

where $\|\mathbf{x}\|$ denotes the Euclidean norm of the projection of $\mathbf{x}$ onto the plane $\{z = 0\}$. Then, for some constant C

$$\Gamma(\log h, \log h)(\mathbf{x}) \le C(1 + d(\mathbf{x})) \tag{13}$$

and

$$|Z(\log h)| \le C. \tag{14}$$

The last one is not completely explicited in [29] but follows easily from the estimation of W_1 page 376 of this paper.

Lemma 2.1. *The Schwartz space $\mathcal{S}$ of smooth rapidly decreasing functions on the Heisenberg group $\mathbb{H}$ is left globally stable by L and by P_t for any $t \ge 0$.*

Proof. If $R = (x^2+y^2)^2+z^2$, then an elementary computation shows that for any positive integer q, there exists a real constant $B_q > 0$ such that $L((1+R)^{-q}) \le B_q(1+R)^{-q}$. As a consequence, $P_t((1+R)^{-q}) \le e^{B_q t}(1+R)^{-q}$. We may see the class $\mathcal{S}$ as the class of smooth functions such that for any non negative integers a, b, c and any positive integer q, the function $\hat{X}^a\hat{Y}^b Z^c(f)$ is bounded above by $(1+R)^{-q}$. From that and the above, it is clear that if f is in $\mathcal{S}$, such is $P_t f$. On the other hand, the stability of $\mathcal{S}$ by L is straightforward. □

3 Reverse Poincaré inequalities

We show here how to deduce a reverse Poincaré inequality as (5). The method is simple and direct, and does not rely on a gradient bound such as (3) or (6).

Theorem 3.1 (Reverse local Poincaré inequality). *For any $t \geq 0$ and any $f \in \mathcal{C}_c^\infty(\mathbb{H})$,*

$$t\Gamma(P_t f, P_t f) \leq P_t(f^2) - (P_t f)^2.$$

Proof. Since we work on a group, it is enough to prove this for $\mathbf{x} = 0$. Then, thanks to the dilation properties, it is enough to prove it for $t = 1$. Now, consider the vector field $\hat{X}$, which coincides with X at $\mathbf{x} = 0$, and let as before h be the density of the heat kernel for $t = 1$ and $\mathbf{x} = 0$ given by (10). We want to bound, for a smooth compactly supported function f

$$(\hat{X} P_1 f)^2 = (P_1(\hat{X} f))^2 = \langle f, \hat{X} h\rangle^2$$

where the last identity comes from integration by parts. The first remark is that we may suppose that $\langle fh\rangle = 0$ since we may always add any constant to f. We then use Cauchy-Schwartz inequality under the measure $h(\mathbf{x})d\mathbf{x}$ to get

$$(\hat{X} P_1 f)^2 \leq P_1(f^2)\langle \hat{X}(\log h)^2 h\rangle.$$

Using the same method for Y, we get

$$\Gamma(P_1 f, P_1 f) \leq (P_1(f^2) - (P_1 f)^2)\langle \hat{\Gamma}(\log h, \log h) h\rangle$$

where

$$\hat{\Gamma}(u, u) = \hat{X}(u)^2 + \hat{Y}(u)^2.$$

Now, the rotational invariance of h, which comes from $[\Theta, L] = 0$, shows that $\hat{\Gamma}(\log h) = \Gamma(\log h)$, and gives a reverse local Poincaré inequality with the constant $C = \langle \Gamma(\log h) h\rangle$. It remains to compute this constant. For that, we use the dilation operator D and the formula (9). In 0, we have $Df = 0$, and therefore it reads for $t = 1$, for any f,

$$P_1((L - D)f) = 0,$$

which means that h is the invariant measure for the operator $L - D$, or in other words that

$$(L + D + 2)h = 0$$

since the adjoint of D is $-D - 2$. Multiply both sides by $\log h$ and using integration by parts (can be rigorously justified by using estimates on h) gives

$$\langle \log h, Lh\rangle = -\langle \Gamma(\log h, \log h), h\rangle.$$

Moreover, we have

$$\langle \log h, (D + 2)h\rangle = -\langle h, D\log h\rangle = -\langle Dh\rangle = 2\langle h\rangle = 2$$

and therefore

$$\langle \Gamma(\log h, \log h), h\rangle = 2.$$

If we had done the same reasoning on $\mathbb{R}^n$ with the usual Laplace operator, and the corresponding dilation operator, we would have find a reverse Poincaré inequality with constant $n/2$ instead of $\frac{1}{2}$. The reason comes from symmetry properties and the same will allow us to divide the constant by 2 in the Heisenberg case. In fact, because of the rotational invariance of h, we have, for any vector $(a, b) \in \mathbb{R}^2$ with $a^2 + b^2 = 1$,

$$\langle (a\hat{X}(\log h) + b\hat{Y}(\log h))^2, h\rangle = \langle (\hat{X}(\log h))^2, h\rangle = \frac{1}{2}\langle \hat{\Gamma}(\log h, \log h), h\rangle = 1.$$

Now, we may write

$$\Gamma(P_1 f, P_1 f)(0) = \sup_{a^2+b^2=1} (a\hat{X}P_1(f) + b\hat{Y}P_1(f))^2$$

and use the same Cauchy-Schwarz inequality to improve the bound to

$$\Gamma(P_1 f, P_1 f) \leq P_1(f^2) - (P_1 f)^2.$$

□

Remark 3.2 (Optimal constants). *Equality is achieved in theorem* (3.1) *when* $f = \hat{X}(\log h)$ *for instance. To see it, note that by symmetry,* $\langle(\hat{X}\log h)^2 h\rangle = \langle(\hat{Y}\log h)^2 h\rangle = 1$*. By the rotational invariance of* h*, we get more generally that* $\langle(a\hat{X}\log h + b\hat{Y}\log h)^2 h\rangle = 1$ *for any* $a^2 + b^2 = 1$*. As a consequence,* $\langle\hat{X}(\log h)\hat{Y}(\log h), h\rangle = 0$*. For* $f = \hat{X}(\log h)$*, this gives* $XP_1 f(0) = \langle\hat{X}f, h\rangle = -\langle f, \hat{X}(\log h)h\rangle = -1$ *and* $YP_1 f = 0$*, which is the desired equality. Note the difference with the elliptic case: for the heat semigroup* $(P_t)_{t\geq 0}$ *in* $\mathbb{R}^n$ *or for any manifold with non negative Ricci curvature, one has for every* $t \geq 0$ *and any smooth* f*,*

$$2t\Gamma(P_t f, P_t f) \leq P_t(f^2) - (P_t f)^2.$$

Note also, as we already mentioned in the introduction, that the H.-Q. Li gradient bound (6) *provides simply by semigroup interpolation a result similar to theorem 3.1, with a constant* $2tC_1^{-2}$ *instead of* t*. These two constants are equal if and only if* $C_1 = \sqrt{2}$*. At the level of reverse local Poincaré inequalities, a necessary and sufficient condition for the efficiency of the semigroup interpolation technique is that* $C_1 = \sqrt{2}$*. Similarly, by using the Drivier and Melcher gradient bound* (3) *we obtain the condition* $C_2 = 2$*. It is thus tempting to conjecture that* $C_1^2 = C_2 = 2$*.*

Remark 3.3 (Carnot groups). *In the class of nilpotent groups, there is an interesting subclass, which are the Carnot groups, that is the nilpotent groups with dilations, see [7], [20]. Let* $(X_1, \ldots, X_{n_0})$ *the generators at the first level of such a Carnot group, and* $L = X_1^2 + \cdots + X_{n_0}^2$*. In those groups again there is a dilation operator* D *such that* $[L, D] = L$ *and* $D^* = -D - \frac{n}{2}Id$*. The parameter* n *is what is called the homogeneous dimension of the group. The same applies in this case, except that we no longer have always enough rotations to insure that* $\Gamma(h,h) = \hat{\Gamma}(h,h)$ *where the hat corresponds to the "chiral" action. But we may replace this argument by the fact that* $\hat{P}_t(\cdot)(0) = P_t(\cdot)(0)$ *where* $(\hat{P}_t)_{t\geq 0} = (\exp(t\hat{L}))_{t\geq 0}$ *and we would get as a bound*

$$\Gamma(P_t f, P_t f) \leq \frac{n}{2n_0 t}(P_t(f^2) - (P_t f)^2)$$

recovering at the same time the Heisenberg and Euclidean cases. For the $(2p+1)$*-dimensional Heisenberg group* $\mathbb{H}_{2p+1}$ *we have* $n_0 = 2p$ *while the homogeneous dimension is* $2p+2$*, and therefore here the inequality writes*

$$\Gamma(P_t f, P_t f) \leq \frac{p+1}{2pt}(P_t(f^2) - (P_t f)^2)$$

and the constant approaches the Euclidean one when p *goes to infinity.*

4 A proof of the Driver-Melcher inequality

We give here an elementary proof of the Driver and Melcher gradient bound (3). The argument is simply an integration by parts followed by the upper bound on $\Gamma(\log h, \log h)$ obtained in section

3. Indeed, from the inequalities (12) and (13), it is quite clear that the constant

$$A = \int \|\mathbf{x}\| \Gamma(\log h, \log h)(\mathbf{x}) h(\mathbf{x}) d\mathbf{x} \tag{15}$$

is finite, where $\|\mathbf{x}\|$ denotes as usual the Euclidean norm of the horizontal projection of the point $\mathbf{x}$. Then, we have the following theorem.

Theorem 4.1. *With the constant A defined by* (15)*, we have for every $t \geq 0$ and $f \in \mathcal{C}_c^\infty(\mathbb{H})$,*

$$\Gamma(P_t f, P_t f) \leq 2(A+4) P_t(\Gamma(f,f)).$$

Proof. We assume that $\mathbf{x} = 0$ (by group action) and $t = 1$ (by dilation). Then, we write

$$X P_1 f(0) = P_1(\hat{X} f)(0) = \langle (X + yZ) f h \rangle.$$

An integration by parts for $\langle yZ(f), h \rangle = \langle y(XY - YX)(f), h \rangle$ gives

$$\langle X(f), (yY(\log h) + 1)h \rangle - \langle Y(f), yX(\log h) h \rangle$$

and a similar formula holds for $Y P_1 f$. Next, we take a vector $(a,b) \in \mathbb{R}^2$ of unit norm and we use the Cauchy-Schwarz inequality to get

$$(aXP_1(f)(0) + bYP_1(f)(0))^2 \leq P_1(X(f)^2) A_1 + P_1(Y(f)^2) A_2$$

where

$$A_1 = P_1[((yY(\log h) + 2)a - xY(\log h) b)^2]$$

and

$$A_2 = P_1[((xX(\log h) + 2)b - yY(\log h) a)^2].$$

The desired inequality comes then from the upper bound

$$\max(A_1, A_2) \leq A_1 + A_2 \leq 2(A+4).$$

Note that the obtained constant $2(A+4)$ is certainly not the optimal one. □

Remark 4.2 (Counter example). *Unlike the elliptic case, the reverse local Poincaré* (5) *and the local Poincaré* (4) *inequalities are not in general equivalent. A simple example is provided on $\mathbb{R}^2$ with the operator*

$$L = \partial_x^2 + x\partial_y$$

for which the corresponding diffusion process starting from (x,y) is up to some constant

$$U_t = \left(x + B_t,\ y + tx + \int_0^t B_s ds \right)$$

where $(B_t)_{t \geq 0}$ is a Brownian motion on $\mathbb{R}$. In this example, the heat kernel is Gaussian and the semigroup $(P_t)_{t\geq 0}$ is quite easy to compute, while $\Gamma(f,f) = (\partial_x f)^2$. In this situation, it is easy to see, using Cauchy-Schwarz inequality, that

$$tC\Gamma(P_t f, P_t f) \leq P_t(f^2) - (P_t f)^2$$

for some constant $C < 2$, while the inequality

$$P_t f^2 - (P_t f)^2 \leq C(t) P_t \Gamma(f,f)$$

does not hold for any constant $C(t)$, as one may see with a function f depending on y only. For example, with $f(x,y) = y$, one has $P_t f = y + tx$, which depends on the variable x. This kind of hypoelliptic situation differs strongly from the case of the Heisenberg group, since here $\Gamma(f,f) = 0$ does not imply that f is constant.

5 Two proofs of the H.-Q. Li inequality

In this section, we propose two alternate and independent proofs of the H.-Q. Li inequality (6). The first proof uses some basic symmetry considerations and a particular case of the Cheeger inequality of theorem 6.3 that we have to show by hands. The second proof relies on an explicit commutation between the complex gradient and the heat semigroup. Both mainly rely on the previous sharp estimates on the heat kernel that were obtained in [19].

5.1 Via a Cheeger type inequality

Lemma 5.1. *For any real $R > 0$, there exists a real constant $C > 0$ such that for any smooth $f : \mathbb{H} \to \mathbb{R}$ which vanishes on the ball centered at 0 and of radius R for the Carnot-Carathéodory distance, we have*

$$\int |f|\, h d\mathbf{x} \le C \int |\nabla f|\, h d\mathbf{x}$$

where h is as before the density of $P_1(0, d\mathbf{x})$.

Proof. One may safely assume that $R = 1$ by a simple scaling. Next, we make use of the polar coordinates which appear in (11). Namely, we parameterize the exterior of the unit ball by (u, s), with $u \in \mathbb{C}$, $|u| \ge \frac{1}{2\pi}$ and $s \in (1, 2\pi |u|)$, with

$$(x + iy, z) = \left(u \left(1 - \exp\left(\frac{is}{|u|} \right) \right), \frac{|u|^2}{2} \left(\frac{s}{|u|} - \sin\left(\frac{s}{|u|} \right) \right) \right). \tag{16}$$

The unit ball is the set $\{s \le 1\}$, and since f is supported outside the unit ball, we write

$$|f(u, s)| = \left| \int_1^s \nabla f(u, t) \cdot e_t dt \right| \le \int_1^s |\nabla f|\,(u, t) dt$$

where e_t is the unit vector along the geodesic. Let us write $A(u, t)dudt$ the Lebesgue measure on $\mathbb{R}^3$ in those coordinates (we shall see the precise formula below). We write

$$\int |f(u, s)|\, h(u, s) A(u, s) duds \le \int |\nabla f|\,(u, t) \left(\int_t^{2\pi|u|} A(u, s) h(u, s) ds \right) dudt$$

and we shall have proved our inequality when we have proved that

$$\int_t^{2\pi|u|} A(u, s) h(u, s) ds \le C A(u, t) h(u, t),$$

for any (u, t) such that $|u| \ge \frac{1}{2\pi}$ and $t \ge 1$. In this computation, we forget the points in the (x, y) plane and the z-axis, but this is irrelevant since they have 0-measure. The computation of the Jacobian gives

$$A(u, s) = 16\, |u| \sin\left(\frac{s}{2\,|u|} \right) \left(\frac{s}{2\,|u|} - \sin\left(\frac{s}{2\,|u|} \right) \right)$$

and the estimate (12) shows that we may replace $h(u, s)$ by

$$\frac{\exp(-\frac{s^2}{4})}{\sqrt{1 + 2s\,|u| \sin(\frac{s}{2|u|})}}$$

since the Euclidean norm of the horizontal projection of the point whose coordinates are (u,s) is $2|u|\sin(\frac{s}{2|u|})$. Setting $\tau = \frac{s}{2|u|}$ and $r = |u|$, the question is therefore to check that, for some constant C, for any $r \geq \frac{1}{2\pi}$, for any $\tau_0 \geq \frac{1}{2r}$, one has

$$r\int_{\tau_0}^{\pi} \frac{\sin\tau(\tau - \sin\tau)}{\sqrt{1+4r^2\tau\sin\tau}} e^{-\tau^2 r^2} d\tau \leq C\frac{\sin\tau_0(\tau_0 - \sin\tau_0)}{\sqrt{1+4r^2\tau_0\sin\tau_0}} e^{-\tau_0^2 r^2}.$$

Up to some constant, we may replace $\sin\tau(\tau - \sin\tau)$ by τ^4 on $(0,\frac{\pi}{2})$ and by $\pi - \tau$ on $(\frac{\pi}{2},\pi)$. In the same way, we may replace $\sqrt{1+4r^2\tau\sin\tau}$ by $r\tau$ when $\tau < \frac{\pi}{2}$ (since $r\tau \geq \frac{1}{2}$) and by $1 + r\sqrt{\pi - \tau}$ when $\tau \in (\frac{\pi}{2},\pi)$.

We first consider the case where $\tau_0 < \frac{\pi}{2}$, and divide the integral into $\int_{\tau_0}^{\pi/2}$ and $\int_{\pi/2}^{\pi}$. Using the above estimates, these integrals can be bounded by the correct term by using the fact that

$$\int_A^{\infty} s^p \exp(-s^2) ds \leq C_p A^{p-1}\exp(-A^2).$$

When $\tau_0 > \frac{\pi}{2}$, one uses the same estimates, bounding above $\pi - \tau$ by $\pi - \tau_0$ and $(1 + r\sqrt{\pi - \tau})^{-1}$ by 1 in the integral, and using the fact that r is bounded below on our domain.

Observe that the same reasoning on a ball of radius ϵ would provide a constant which goes to infinity when ϵ goes to 0, as for the usual heat kernel on $\mathbb{R}^d$. □

In fact, we shall also use a slightly improved version of lemma 5.1.

Lemma 5.2. *For every real $R > 0$, if B is the ball centered at 0 and of radius R for the Carnot-Carathéodory distance, there exists a real constant $C > 0$ such that for any smooth $f : \mathbb{H} \to \mathbb{R}$,*

$$\int_{B^c} |f - m|\, h d\mathbf{x} \leq C \int |\nabla f|\, h d\mathbf{x}$$

where $B^c = \mathbb{H}\setminus B$ is the complement of B, where $m = |B|^{-1}\int_B f(x)\, d\mathbf{x}$ is the mean of f on B, and where h is as before the density of $P_1(0, d\mathbf{x})$.

For proving this last lemma, we will need the following L^1-Poincaré, also called $(1,1)$ Poincaré, on balls. This inequality can be in fact thought of as a Cheeger type inequality on balls. See [32] and references therein. This last lemma shall also be used in the next section where we prove the H.-Q. Li inequality via complex analysis.

Lemma 5.3. *For any real $R > 0$, if B denotes the ball centered at 0 and of radius R for the Carnot-Carathéodory distance, there exists a real constant $C > 0$ such that for any smooth $f : \mathbb{H} \to \mathbb{R}$, by denoting $m = |B|^{-1}\int_B f(x)\, d\mathbf{x}$ the mean of f on B,*

$$\int_B |f(x) - m|\, d\mathbf{x} \leq C \int_B |\nabla f|\,(x)\, d\mathbf{x}.$$

We can now make the proof of lemma 5.2.

Proof of lemma 5.2. As in lemma 5.1, we may safely assume that $R = 1$ by a simple scaling. For any auxillary function $g : \mathbb{H} \to \mathbb{R}$, we have by denoting $m = |B|^{-1}\int_B f\, d\mathbf{x}$,

$$\int_{B^c} |f - m|\, h\, d\mathbf{x} \leq \int |f - g|\, h\, d\mathbf{x} + \int_{B^c} |g - m|\, h\, d\mathbf{x}.$$

Now we choose g such that $g(\xi, s) = f(\xi, \min(s, 1))$ where (ξ, s) denotes the polar coordinates in $\mathbb{H}$ as in the proof of lemma 5.1. More precisely, $\xi \in \partial B$ and the set $\{s \leq 1\}$ is the unit ball. For the first term the desired gradient bound follows then by elementary arguments as in lemma 5.1. For the second term, we write

$$|f(\xi, 1) - m| \leq \int_{s=0}^{1} \left(|f(\xi, 1) - f(\xi, s)| + |f(\xi, s) - m|\right) \frac{A(\xi, s)ds}{C(\xi)}$$

where $C(\xi) = \int_{s=0}^{1} A(\xi, s)ds$. We can now conclude by using elementary arguments similar as before and the L^1-Poincaré inequality of lemma 5.3. □

Note that lemma 5.1 can be deduced directly from lemma 5.2. We are now in position to prove the H.-Q. Li inequality (6).

Proof of (6). With the help of lemmas 5.1 and 5.2, we may reduce the study of the H.-Q. Li inequality to functions which are

- either supported in a ball of radius 1 for the Carathéodory metric;
- either supported in a cylinder of radius 2 around the z axis (without the unit ball);
- either supported outside a cylinder around the z-axis.

Indeed, let see how one may reduce first to the case of a function supported either in a ball or outside a ball. If f is any smooth function and ϕ a smooth cutoff function with values 1 on a ball B of radius < 1 and vanishing outside a ball of radius 1, we write $f = f\phi + f(1 - \phi) = f_1 + f_2$. Clearly, in order to obtain (6), one can add any prescribed constant to f. In particular, one can assume that $\int_B f \, d\mathbf{x} = 0$. Assuming that we know the inequality for f_1 and f_2, we bound

$$\langle \hat{X}(f), h \rangle \leq C \langle (|\nabla f_1| + |\nabla f_2|), h \rangle$$

then we make use of

$$|\nabla f_1| + |\nabla f_2| \leq |\nabla f| + 2\,|f|\,|\nabla \phi|$$

and since $|\nabla \phi|$ is supported outside the unit ball,

$$|f|\,|\nabla \phi| \leq ||\nabla \phi||_\infty |f| 1_{B^c}$$

so one has by lemma 5.2

$$\langle |f|\,, |\nabla \phi|\, h \rangle \leq C \langle |\nabla f|\,, h \rangle.$$

We repeat the same operation with a cutoff function for the neighborhood of the z-axis. Now, when f is supported inside the ball, we may use the method that we used in the proof of theorem 4.1, and the fact that $|\nabla \log h|\,(\mathbf{x}) \leq C d(\mathbf{x})$, which is bounded on the unit ball. If f is supported inside the cylinder around the z-axis and vanishes on the unit ball, we write, with section 2 notations,

$$\langle \hat{X}(f), h \rangle = \langle X(f), h \rangle + \langle f, \frac{y}{2} Z(\log h) h \rangle$$

and then we use the fact that $\frac{y}{2} Z(\log h)$ is bounded on the cylinder. It remains to observe that

$$\langle |f|\,, h \rangle \leq C \langle |\nabla f|\,, h \rangle$$

thanks to lemma 5.1. It remains to deal with a function which is supported outside a cylinder around the z-axis. We shall choose another integration by parts. For that, let us use a complex notation and write

$$\nabla(f) = X(f) + iY(f) \quad \text{and} \quad \hat{\nabla}(f) = \hat{X}(f) - i\hat{Y}(f).$$

Note the change of sign in front of i in the second expression. We want to bound

$$\langle \hat{\nabla}(f), h\rangle = -\langle f, \hat{\nabla} h\rangle.$$

Now, since h is radial, we have

$$\hat{\nabla} h = \frac{x - iy}{x + iy} \nabla h$$

which comes from the fact that $x\partial_y h = y\partial_x h$. Let us call $\Psi(x, y) = \exp(-2i\theta)$ the function $\frac{x-iy}{x+iy}$, where θ is the angle in the plane (x, y). Then, we integrate again by parts and get

$$\langle \hat{\nabla} f, h\rangle = -\langle f, \Psi(x, y)\nabla h\rangle = \langle \nabla f, \Psi(x, y)h\rangle + \langle f, \nabla(\Psi)h\rangle.$$

We then conclude observing that Ψ is bounded and $|\nabla\Psi|$ is bounded outside the cylinder around the z axis. We therefore have

$$\left|\langle \hat{\nabla}(f), h\rangle\right| \leq \langle |\nabla f|, h\rangle + C\langle |f|, h\rangle$$

and we use again lemma 5.2 to conclude the proof. □

5.2 Via a complex quasi-commutation

In $\mathbb{R}^n$, it is known that the gradient ∇ commute with the Laplace operator. This commutation leads to the commutation between ∇ and the heat semigroup $P_t = e^{t\Delta}$ and therefore to the inequality:

$$|\nabla P_t f| = |P_t \nabla f| \leq P_t |\nabla f|.$$

In the Heisenberg group, we can follow the same pattern of proof. Nevertheless, several difficulties appear that make the proof quite delicate and technical at certain points. For sake of clarity, before we enter the hearth of the proof, let us precise our strategy. The Lie algebra structure:

$$[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0$$

leads to the commutation:

$$(X + iY)L = (L - 2iZ)(X + iY),$$

where $L = X^2 + Y^2$. At the level of semigroups, it leads to the *formal* commutation:

$$(X + iY)P_t = e^{t(L-2iZ)}(X + iY) = e^{-2itZ}P_t(X + iY). \tag{17}$$

This commutation is only formal because as we will see the semigroup associated to the complex operator $L - 2iZ$ is not globally well defined. More precisely, complex solutions to the heat equation $\frac{\partial u}{\partial t} = (L - 2iZ)u$, $u(0, \cdot) = f$ are represented by a kernel which is nothing esle than the holomorphic complex extension in the z variable of the heat kernel, at the point $z + 2it$. Unfortunately, this kernel has poles, and this solution may have singularities. Nevertheless, we will see that if the initial condition f is a complex gradient, then solutions to this equation do not explode. More precisely, we may add to this kernel any kernel which has no effect on gradients and which cancels the poles of the previous extension. Doing this, we shall produce an integral representation of the solution, without poles. This representation is of course not unique. If we could choose the kernel in such a way that the ratio of it with the density p_t is bounded, then the H.-Q. Li inequality would easily follow. However, we will prove that such a kernel does not exist. To overcome this difficulty, we will use two different kernels depending on the support of the function f. By using a partition of the unity as in our previous proof of H.-Q. Li inequality and lemma 5.3 we will then be able to conclude.

We now enter into the hearth of the proof. In what follows, in order to exploit the rotational invariance, we shall use the cylindric coordinates $x = r\cos\theta$, $y = r\sin\theta$ in which the vector fields X and Y read

$$X = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta - \frac{1}{2}r\sin\theta\partial_z$$
$$Y = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta + \frac{1}{2}r\cos\theta\partial_z$$
$$Z = \partial_z.$$

The heat kernel associated to $(P_t)_{t\geq 0}$ writes here in cylindric coordinates

$$p_t(r,z) = \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} e^{i\lambda z}\frac{\lambda}{\sinh\lambda t}e^{-\frac{r^2}{4}\lambda\mathrm{cotanh}\lambda t}d\lambda. \tag{18}$$

To give a sense to (17), we begin with the analytic properties of $p_t(r,z)$ in the variable z.

Lemma 5.4. *Let $t > 0$ and $r \geq 0$. The function*

$$z \to p_t(r,z) - \frac{1}{4\pi^2\left(t + iz + \frac{r^2}{4}\right)^2} - \frac{1}{4\pi^2\left(t - iz + \frac{r^2}{4}\right)^2}$$

admits an analytic continuation on $\left\{z \in \mathbb{C}, \mid \mathbf{Im} z \mid < \frac{r^2}{4} + 3t\right\}$. The function

$$z \to p_t(r,z)$$

admits therefore a meromorphic continuation on $\left\{z \in \mathbb{C}, \mid \mathbf{Im} z \mid < \frac{r^2}{4} + 3t\right\}$ with double poles at $-i\left(t + \frac{r^2}{4}\right)$ and $i\left(t + \frac{r^2}{4}\right)$.

Proof. Let $t > 0$ and $r \geq 0$. By using the expression (18) for $p_t(r,z)$, and

$$\frac{1}{\left(t + iz + \frac{r^2}{4}\right)^2} = \int_0^{+\infty} e^{-i\lambda z}e^{-\lambda t}e^{-\lambda\frac{r^2}{4}}\lambda d\lambda,$$
$$\frac{1}{\left(t - iz + \frac{r^2}{4}\right)^2} = \int_0^{+\infty} e^{i\lambda z}e^{-\lambda t}e^{-\lambda\frac{r^2}{4}}\lambda d\lambda,$$

we obtain

$$p_t(r,z) - \frac{1}{4\pi^2\left(t + iz + \frac{r^2}{4}\right)^2} - \frac{1}{4\pi^2\left(t - iz + \frac{r^2}{4}\right)^2}$$
$$= \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} e^{i\lambda z}\left(\frac{e^{-\frac{r^2}{4}|\lambda|\mathrm{cotanh}|\lambda|t}}{\sinh\mid\lambda\mid t} - 2e^{-\frac{1}{4}|\lambda|r^2-|\lambda|t}\right)\mid\lambda\mid d\lambda$$

and the desired result follows easily. $\square$

For any $t > 0$, $r \geq 0$, and $z \in \mathbb{C} - \{-i(t + \frac{1}{4}r^2)\}$ such that $|\, \mathbf{Im} z \,| < \frac{r^2}{4} + 3t$, let us denote

$$p_t^*(r,z) = p_t(r,z) - \frac{1}{4\pi^2 \left(t + iz + \frac{r^2}{4}\right)^2}.$$

We have the following commutation property.

Proposition 5.5. *If $f : \mathbb{H} \to \mathbb{R}$ is a smooth function with compact support, then*

$$(X + iY)P_t f(0) = \int_{\mathbb{H}} p_t^*(r, z + 2it)(X + iY)f(r,\theta,z) r dr d\theta dz, \quad t > 0.$$

Proof. Due to the identities $[X,Y] = Z$ and $[X,Z] = [Y,Z] = 0$, we have

$$(X + iY)L = (L - 2iZ)(X + iY).$$

If $f(r,\theta,z) = e^{i\lambda z} g(r,\theta)$, for some $\lambda \in \mathbb{R}$ and some function g, we have $Zf = i\lambda f$ and thus

$$(X + iY)Lf = (L + 2\lambda)(X + iY)f.$$

We deduce therefore,

$$(X + iY)P_t f(0) = e^{2\lambda t}(P_t(X + iY)f)(0) = e^{2\lambda t} \int_{\mathbb{H}} p_t(r,z)((X + iY)f)(r,\theta,z) r dr d\theta dz.$$

Let us now observe that for $t > 0$,

$$(X + iY)\frac{1}{\left(t + iz + \frac{r^2}{4}\right)^2} = 0$$

and thus

$$(X + iY)p_t^* = (X + iY)p_t.$$

Consequently,

$$(X + iY)P_t f(0) = e^{2\lambda t} \int_{\mathbb{H}} p_t^*(r,z)((X + iY)f)(r,\theta,z) r dr d\theta dz.$$

Now

$$e^{2\lambda t} f(r,\theta,z) = f(r,\theta,z - 2it)$$

and the result for the function f follows by integrating by parts with respect to the variable z. For general f, we can conclude by using the Fourier transform of f with respect to the variable z. □

As a first consequence, we deduce that for every $R > 0$, there exists a finite constant $C > 0$ such that for every smooth function compactly supported inside a Carnot-Carathéodory ball $\mathbf{B}_R$ of radius R,

$$|\nabla P_1 f|(0) \leq C P_1(|\nabla f|)(0).$$

But of course, here, the constant C that we obtain depends on R, and we shall see below that it blows up when $R \to +\infty$.

Now, if $R > 0$ is big enough, the ball with radius R contains the region of the Heisenberg group whose cylindric coordinates are of the form $(r = 2, \theta \in [0, 2\pi], z = 0)$ and if f is a smooth function with compact support that vanishes in a ball with radius R, we have the commutation:

$$(X + iY)P_1 f(0) = \int_{\mathbb{H}} p_1(r, z + 2i)(X + iY)f(r,\theta,z) r dr d\theta dz, \quad t > 0.$$

that follows from the fact that $(X + iY)p_t = (X + iY)p_t^*$ and from the fact that the pole of $(r,z) \to p_1(r,z)$ is at $r = 2$, $z = 0$. The keypoint is then the following estimate:

Proposition 5.6. *There exists $R > 0$ such that*

$$\sup_{r^2+|z|\geq R} \frac{|\ p_1(r, z+2i)\ |}{p_1(r,z)} < +\infty.$$

Proof. We shall proceed in two steps.

Step 1. We show that for any $\eta > 0$,

$$\sup_{r\geq 3, r^2\geq \eta |z|} \frac{|p_1(r, z+2i)|}{p_1(r,z)} < +\infty.$$

For convenience, and by symmetry, we may assume $z > 0$. Let us first observe that on our domain:

$$p_1(r, z+2i) = \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} e^{-2\lambda} e^{i\lambda z} \frac{\lambda}{\sinh\lambda} e^{-\frac{r^2}{4}\lambda \mathrm{cotanh}\lambda} d\lambda \tag{19}$$

¿From [19], it is known that for fixed r, z, the function

$$g : \lambda \to -i\lambda z + \frac{r^2}{4}\lambda \mathrm{cotanh}\lambda,$$

has a unique critical point in the strip $\{|\ \mathbf{Im}\lambda\ | < \frac{\pi}{2}\}$. This critical point is $i\theta(r,z)$, where $\theta(r,z)$ the unique solution in $(0, \frac{\pi}{2})$ of the equation

$$\mu(\frac{1}{2}\theta(r,z))r^2 = 4z,$$

with $\mu(\theta) = \frac{\theta}{\sin^2\theta} -$ cotan θ. At this critical point, we have

$$g(i\theta(r,z)) = \frac{d^2(r,z)}{4},$$

where $d(r,z)$ is the Carnot-Carathéodory distance from 0 to the point with cylindric coordinates (r,θ,z) (this distance does not depend on θ, that is why it is omitted in the notation). In fact, our function g corresponds to $g(r,z,\lambda) = f(\frac{r}{\sqrt{2}}, \frac{z}{2}, 2\lambda)$ where f is the function studied in [19].

Moreover the function $s \to \mathbf{Re}g(s + i\theta(r,z))$, grows with $|\ s\ |$, and has a global minimum at $s = 0$, indeed a tedious computation shows that

$$\begin{aligned}
\mathbf{Re}(g(s+i\theta(r,z)) - g(i\theta(r,z))) &= \frac{\sinh^2 2s}{\sinh^2 2s + \sin^2 2\theta(r,z)}(2s \text{ cotanh } 2s - 2\theta(r,z) \text{ cotan } 2\theta(r,z))r^2 \\
&\geq \frac{\sinh^2 2s}{\sinh^2 2s + 1}(2s \text{ cotanh } 2s - 1)r^2 \\
&\geq 0.
\end{aligned}$$

Let us finally observe that the previous computation also shows that there exists $\delta > 0$ such that for $s \in [-1, 1]$

$$\mathbf{Re}g(s + i\theta(r,z)) \geq \frac{d^2(r,z)}{4} + \delta r^2 s^2.$$

With all this in hands, we can now turn to our proof. We first start by changing the contour of integration in (19):

$$\begin{aligned}
\int_{-\infty}^{+\infty} e^{-2\lambda} e^{i\lambda z} \frac{\lambda}{\sinh\lambda} e^{-\frac{r^2}{4}\lambda \mathrm{cotanh}\lambda} d\lambda &= \int_{\mathbf{Im}\lambda = \theta(\sqrt{r^2-8}, z)} e^{-2\lambda} e^{i\lambda z} \frac{\lambda}{\sinh\lambda} e^{-\frac{r^2}{4}\lambda \mathrm{cotanh}\lambda} d\lambda \\
&= \int_{\mathbf{Im}\lambda = \theta(\sqrt{r^2-8}, z)} e^{i\lambda z} \frac{\lambda}{\sinh\lambda} e^{-\left(\frac{r^2}{4}-2\right)\lambda \mathrm{cotanh}\lambda} e^{2\lambda - 2\lambda \mathrm{cotanh}\lambda} d\lambda
\end{aligned}$$

Therefore, by denoting

$$U(\lambda) = e^{2\lambda - 2\lambda \text{cotanh}\lambda} \frac{\lambda}{\sinh \lambda}$$

we get

$$\begin{aligned}\left| \int_{-\infty}^{+\infty} e^{-2\lambda} e^{i\lambda z} \frac{\lambda}{\sinh \lambda} e^{-\frac{r^2}{4}\lambda \text{cotanh}\lambda} d\lambda \right| \le & e^{-\frac{d(\sqrt{r^2-8},z)^2}{4}} \int_{|s|\le 1} e^{-(r^2-8)\delta^2 s^2} \left| U(s+i\theta(\sqrt{r^2-8},z)) \right| ds \\ & + e^{-\frac{d(\sqrt{r^2-8},z)^2}{4}} \int_{|s|\ge 1} e^{-(r^2-8)\delta^2} \left| U(s+i\theta(\sqrt{r^2-8},z)) \right| ds \\ \le & C_1 \frac{e^{-\frac{d(r,z)^2}{4}}}{r},\end{aligned}$$

where we used the fact that on the domain on which we work, the difference $d(\sqrt{r^2-8},z) - d(r,z)$ is uniformly bounded. Finally, from the lower estimate of [30], on the considered domain,

$$p_t(r,z) \ge C_2 \frac{e^{-\frac{d(r,z)^2}{4}}}{r}.$$

It concludes the proof of step 1.

Step 2. We show that there exists $\eta > 0$ such that

$$\sup_{|z|\ge 1, r^2 \le \eta |z|} \frac{|p_1(r, z+2i)|}{p_1(r,z)} < +\infty.$$

We first start by giving an analytic representation of

$$p_1(r, z+2i)$$

that is valid on the domain on which we work. As in the previous proof, we assume $z > 0$. Due to the Cauchy theorem, we can change the contour of integration in the representation (18), to get with $0 < \varepsilon < \pi$,

$$\begin{aligned} p_1(r,z) &= \frac{1}{8\pi^2} \sum_{k=1}^{+\infty} \int_{|\lambda - ik\pi| = \varepsilon} e^{i\lambda z} \frac{\lambda}{\sinh \lambda} e^{-\frac{r^2}{4}\lambda \text{cotanh}\lambda} d\lambda \\ &= \frac{-i}{8\pi^2} \sum_{k=1}^{+\infty} \int_{|\lambda| = \varepsilon} e^{i(-i\lambda + ik\pi)z} \frac{(-i\lambda + ik\pi)}{\sinh(-i\lambda + ik\pi)} e^{-\frac{r^2}{4}(-i\lambda + ik\pi)\text{cotanh}(-i\lambda + ik\pi)} d\lambda \\ &= \frac{-i}{8\pi^2} \int_{|\lambda|=\varepsilon} \frac{e^{-(\pi-\lambda)\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)}}{1 + e^{-\pi\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)}} \left(\frac{\pi}{1 + e^{-\pi\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)}} - \lambda \right) \frac{d\lambda}{\sin \lambda} \end{aligned}$$

Therefore, for $z > 0$,

$$\begin{aligned} & p_1^*(r, z+2i) + \frac{1}{4\pi^2 \left(-1 + iz + \frac{r^2}{4}\right)^2} \\ =& \frac{-i}{8\pi^2} \int_{|\lambda|=\varepsilon} e^{2i\lambda} \frac{e^{-(\pi-\lambda)\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)}}{1 + e^{-\pi\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)}} \left(\frac{\pi}{1 + e^{-\pi\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)}} - \lambda \right) \frac{d\lambda}{\sin \lambda} \end{aligned}$$

On our domain, if η is small enough, when $r, z \to +\infty$, $\mathbf{Re}(z - \frac{r^2}{4} \text{ cotan } \lambda)$ goes uniformly on the circle $\mid \lambda \mid = \varepsilon$ to $+\infty$. Consequently, on our domain

$$\left| p_1^*(r, z + 2i) + \frac{1}{4\pi^2 \left(-1 + iz + \frac{r^2}{4}\right)^2} \right| \leq c_1 \left| \int_{|\lambda|=\varepsilon} e^{2i\lambda} e^{-(\pi-\lambda)\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)} (\pi - \lambda) \frac{d\lambda}{\sin \lambda} \right|$$

for some finite positive constant c_1. By choosing $\varepsilon = \pi - 2\theta(r, z)$, we have

$$\begin{aligned} &\int_{|\lambda|=\varepsilon} e^{2i\lambda} e^{-(\pi-\lambda)\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)} (\pi - \lambda) \frac{d\lambda}{\sin \lambda} \\ = &\int_{|\lambda|=\pi-2\theta(r,z)} e^{2i\lambda} e^{-(\pi-\lambda)\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)} (\pi - \lambda) \frac{d\lambda}{\sin \lambda}, \end{aligned}$$

where the function $\theta(r, z)$ has been introduced above. At this stage, we can follow step by step the proof of Theorem 2.17 in [19] (the only difference is in the function V which we take equal to $V(\lambda) = e^{2i\lambda} \frac{\pi-\lambda}{\sin \lambda}$) to get an estimate on our domain :

$$\left| \int_{|\lambda|=\pi-2\theta(r,z)} e^{2i\lambda} e^{-(\pi-\lambda)\left(z - \frac{r^2}{4} \text{ cotan } \lambda\right)} (\pi - \lambda) \frac{d\lambda}{\sin \lambda} \right| \leq c_2 \frac{e^{-\frac{d(r,z)^2}{4}}}{\sqrt{rd(r, z)}}$$

for some finite positive constant c_2. Finally, the lower estimate of [30] leads to the conclusion. $\square$

Remark 5.7. *In order to extend the H.-Q. Li inequality to more general situations, it would be interesting to get a proof of the above proposition that would not use the explicit expression for* $p_t(r, z)$.

We can now reprove H.-Q. Li's inequality by using a partition of the unity (which is here simpler than in the previous subsection) and the L^1-Poincaré inequality of lemma 5.3 (which was also used in the previous subsection). Let $f : \mathbb{H} \to \mathbb{R}$ be a smooth positive function compactly supported and let $0 \leq \phi \leq 1$ be a smooth function that takes the value 1 on a ball $\mathbf{B}_{R_1}$ and the value 0 outside the ball $\mathbf{B}_{R_2}$ where $R_1 < R_2$, with R_1 big enough. We have

$$\begin{aligned} (X + iY)P_1 f(0) =&(X + iY)P_1 \phi f(0) + (X + iY)P_1(1 - \phi)f(0) \\ =&\int_{\mathbb{H}} p_1^*(r, z + 2i)(X + iY)(f\phi)(r, \theta, z) r dr d\theta dz \\ &+ \int_{\mathbb{H}} p_1(r, z + 2i)(X + iY)(f(1 - \phi))(r, \theta, z) r dr d\theta dz \\ =&\int_{\mathbb{H}} \phi(r, \theta, z) p_1^*(r, z + 2i)(X + iY)f(r, \theta, z) r dr d\theta dz \\ &+ \int_{\mathbb{H}} (1 - \phi(r, \theta, z)) p_1(r, z + 2i)(X + iY)f(r, \theta, z) r dr d\theta dz \\ &+ \frac{1}{4\pi^2} \int_{\mathbb{H}} f(r, \theta, z) \frac{(X + iY)\phi(r, \theta, z)}{\left(-1 + iz + \frac{r^2}{4}\right)^2} r dr d\theta dz. \end{aligned}$$

Therefore

$$\mid \nabla P_1 f(0) \mid \leq C P_1 \mid \nabla f \mid (0) + \left| \frac{1}{4\pi^2} \int_{\mathbb{H}} f(r, \theta, z) \frac{(X + iY)\phi(r, \theta, z)}{\left(-1 + iz + \frac{r^2}{4}\right)^2} r dr d\theta dz \right|.$$

Now, we estimate $\left|\int_{\mathbb{H}} f(r,\theta,z)\frac{(X+iY)\phi(r,\theta,z)}{\left(-1+iz+\frac{r^2}{4}\right)^2}rdrd\theta dz\right|$ thanks to lemma 5.3:

$$\begin{aligned}
&\left|\int_{\mathbb{H}} f(r,\theta,z)\frac{(X+iY)\phi(r,\theta,z)}{\left(-1+iz+\frac{r^2}{4}\right)^2}rdrd\theta dz\right| \\
=&\left|\int_{\mathbb{H}} (f(r,\theta,z)-m)\frac{(X+iY)\phi(r,\theta,z)}{\left(-1+iz+\frac{r^2}{4}\right)^2}rdrd\theta dz\right| \qquad (m \text{ is the mean of } f \text{ on } \mathbf{B}_{R_2}) \\
\leq& C_1\int_{\mathbf{B}_{R_2}} \mid f(r,\theta,z)-m\mid rdrd\theta dz \\
\leq& C_2\int_{\mathbf{B}_{R_2}} \mid \nabla f\mid (r,\theta,z) rdrd\theta dz \\
\leq& C_3 P_1 \mid \nabla f \mid (0).
\end{aligned}$$

This completes the proof of H.-Q. Li's inequality.

As we mentioned it in the beginning of this section, interestingly, it is not possible to find a function ϕ on $\mathbb{H}$ such that:

- $(X+iY)\phi = 0$;
- The ratio $\frac{|p_1^*(r,z+2i)-\Phi(r,\theta,z)|}{p_1(r,z)}$ is bounded.

Indeed, the first point implies that Φ can be written:

$$\Phi(r,\theta,z) = H\left(\frac{r^2}{4}+iz, re^{i\theta}\right),$$

where $H : \{z_1\in\mathbb{C}, \mathbf{Re}(z_1)\geq 0\}\times\mathbb{C}\to\mathbb{C}$ is analytic in z_1 and z_2. Now, due to the estimate of Proposition 5.6 and the estimate on p_1, it would imply that for r and z, such that $r^2+\mid z\mid$ is big enough:

$$\left|H\left(\frac{r^2}{4}+iz, re^{i\theta}\right)+\frac{1}{4\pi^2\left(-1+iz+\frac{r^2}{4}\right)^2}\right|\leq Ae^{-B(r^2+|z|)}$$

where A and B are strictly positive constants. Now, we have the following lemma that prevents the existence of such H:

Lemma 5.8. *Let $f : \{z_1\in\mathbb{C}, \mathbf{Re}(z_1)\geq 0\}\times\mathbb{C}\to\mathbb{C}$ be analytic in z_1 and z_2. If there exist strictly positive constants A and B such that*

$$\forall r\geq 0,\ \forall z\in\mathbb{R},\ \forall\theta\in[0,2\pi],\quad \left|f\left(r^2+iz, re^{i\theta}\right)\right|\leq Ae^{-B(r^2+|z|)}$$

then $f=0$.

Proof. Let $r\geq 0, z\in\mathbb{R}$. The function $z_2\to f\left(r^2+iz, z_2\right)$ is analytic, therefore from the maximum principle we have

$$\left|f\left(r^2+iz, z_2\right)\right|\leq Ae^{-B(|z_2|^2+|z|)},$$

for $\mid z_2\mid\leq r$. Consequently, on the set $\mathbf{Re}(z_1)\geq\mid z_2\mid^2$ we have

$$\left|f\left(z_1,z_2\right)\right|\leq Ae^{-B(|z_2|^2+|\mathbf{Im}(z_1)|)}.$$

By using the analytic function $z_1 \to f(z_1, z_2)$, a translation, and a multiplication by e^{-z_1}we would therefore obtain a function g analytic on the set $\mathbf{Re}(z) > 0$ such that

$$|g(z)| \leq \alpha e^{-\beta|z|}$$

with $\alpha, \beta > 0$, and such function has clearly to be 0 (Use for instance the conformal equivalence between the set $\mathbf{Re}(z) > 0$ and the open unit disc to get a function h analytic on the disc that satisfy the estimate $|g(z)| \leq \alpha' e^{-\frac{\beta'}{|z|}}$). $\square$

6 Functional inequalities for the heat kernel

Most of the consequences of the classical gradient bounds under a Γ_2 curvature assumption remain true under an H.-Q. Li gradient bound. In the sequel, we derive, by interpolation from the gradient bound (6), several local functional inequalities of Gross-Poincaré-Cheeger-Bobkov type for the heat kernel on the Heisenberg group. The term *local* means that these inequalities concern the probability measure $P_t(\cdot)(\mathbf{x})$ at fixed t and $\mathbf{x}$, in contrast to inequalities for the invariant measure. These local inequalities can be seen as global inequalities for Gaussian measures on the Heisenberg group. In the literature, these inequalities and interpolations where mainly developed in Riemannian settings under a Γ_2 curvature assumption. Rigorously, the semigroup interpolations used in the sequel rely on the existence of an algebra of functions $\mathcal{A}$ from $\mathbb{H}$ to $\mathbb{R}$ stable by the action of the heat kernel. Thanks to lemma 2.1, the Schwartz class $\mathcal{S}$ of smooth and rapidly decreasing functions in $\mathbb{R}^3$ may play this role in the case of the Heisenberg group $\mathbb{H}$.

6.1 Gross-Poincaré type inequalities

One of the first consequence of the gradient bound (6) is Gross-Poincaré type local inequalities, also called φ-Sobolev inequalities in [12, 23]. Namely, let $\varphi : I \to \mathbb{R}$ be a smooth convex function defined on an open interval $I \subset \mathbb{R}$ such that $\varphi'' > 0$ on I and $-1/\varphi''$ is convex on I.

Theorem 6.1 (Local Gross-Poincaré inequalities). *By using the notations of* (6)*, for every* $t \geq 0$*, every* $\mathbf{x} \in \mathbb{H}$*, and every* $f \in \mathcal{C}_c^\infty(\mathbb{H}, I)$*,*

$$P_t(\varphi(f)) - \varphi(P_t f) \leq C_1^2\, t\, P_t\, (\varphi''(f)|\nabla f|^2). \tag{20}$$

Proof. One can assume that the support of f is strictly included in I. Since L is a diffusion operator, $L(\alpha(f)) = \alpha'(f)Lf + \alpha''(f)\Gamma f$ for any $f \in \mathcal{C}_c^\infty(\mathbb{H}, \mathbb{R})$ and any smooth $\alpha : \mathbb{R} \to \mathbb{R}$. By the semigroup and the diffusion properties,

$$P_t(\varphi(f)) - \varphi(P_t f) = \int_0^t \partial_s P_s(\varphi(P_{t-s} f))\, ds = \int_0^t P_s(\varphi''(P_{t-s} f)\, |\nabla P_{t-s} f|^2)\, ds.$$

Now, (6) gives $|\nabla P_{t-s} f|^2 \leq C_1^2 (P_{t-s}(|\nabla f|))^2$. Next, by the Cauchy-Schwarz inequality or alternatively by the Jensen inequality for the bivariate convex function $(u, v) \mapsto \varphi''(u)v^2$, we get $\varphi''(P_{t-s} f)(P_{t-s}(|\nabla f|))^2 \leq P_{t-s}(\varphi''(f)|\nabla f|^2)$, which gives the desired result. $\square$

- for $\varphi(u) = u \log(u)$ on $I = (0, \infty)$, we get a Gross logarithmic Sobolev inequality, mentioned for instance in [29] (see also [21, 22]),

$$P_t(f \log(f)) - P_t(f) \log(P_t(f)) \leq C_1^2\, t\, P_t \left(f^{-1} |\nabla f|^2\right); \tag{21}$$

- for $\varphi(u) = u^p$ on $I = (0, \infty)$ with $1 < p \le 2$, we get a Beckner-Latała-Oleszkiewicz type inequality (see [8, 26])

$$\frac{P_t(f^p) - (P_t(f))^p}{p-1} \le p\, C_1^2\, t\, P_t(f^{p-2}|\nabla f|^2); \tag{22}$$

- for $\varphi(u) = u^2$ on $I = \mathbb{R}$, we get a Poincaré inequality, mentioned in [16],

$$P_t(f^2) - (P_t(f))^2 \le 2\, C_1^2\, t\, P_t(|\nabla f|^2). \tag{23}$$

We have seen in the introduction that a local Poincaré inequality such as (23) can be also obtained from the Driver and Melcher gradient bound (3), with a constant $2C_2$ instead of $2C_1^2$. However, the inequalities (21) and (22) need the stronger gradient bound (6) of H.-Q. Li. They also imply the local Poincaré inequality (23) by linearization. It is shown in [13, Theorem 4.4] that the convexity of the bivariate function $(u, v) \mapsto \varphi''(u)v^2$ is equivalent to the convexity of the φ-entropy functional and also to the tensorization property of the φ-entropy functional. This fact is related to the infinite dimensional nature of (20). The inequality (22) interpolates between (21) (let $p \to 1^+$) and (23) (take $p = 2$). The linearity with respect to t of the constant in front of the right hand side of (20) is related to the fact that $(P_t)_{t\ge0}$ is a convolution semigroup, namely $P_t(\cdot)(\mathbf{x})$ can be obtained from $P_1(\cdot)(0)$ by $\mathbf{x}$-translation and $\sqrt{t}$-dilation in $\mathbb{H}$.

6.2 Cheeger type isoperimetric inequalities

As mentioned in the introduction, it is possible to deduce a reverse local Poincaré inequality from the gradient bounds (3) of Driver and Melcher or (6) of H.-Q. Li. However, the constants are not known precisely. A better constant is provided by theorem 3.1, which implies immediately that for every $t \ge 0$ and every $f \in \mathcal{C}_c^\infty(\mathbb{H}, \mathbb{R})$,

$$\left\| |\nabla P_t f| \right\|_\infty \le \frac{1}{\sqrt{t}} \left\| f \right\|_\infty. \tag{24}$$

Cheeger derived in [14] a lower bound for the spectral gap of the Laplacian on a Riemannian manifold. This bound can be related to a sort of L^1 Poincaré inequality, which has an isoperimetric content, see [15] and references therein. Here we derive such an inequality for the heat kernel by only using the gradient bound (6), by mixing arguments borrowed from [5] and [27].

Theorem 6.2 (Local Cheeger type inequality). *With the notations of* (6)*, for every $t \ge 0$, every $\mathbf{x} \in \mathbb{H}$, and every $f \in \mathcal{C}_c^\infty(\mathbb{H}, \mathbb{R})$,*

$$P_t(|f - P_t(f)(\mathbf{x})|)(\mathbf{x}) \le 4C_1\sqrt{t}\, P_t(|\nabla f|)(\mathbf{x}). \tag{25}$$

Proof. We adapt the method used in [27, p. 953] for the invariant measure in Riemannian settings. For any $g \in \mathcal{C}_c^\infty(\mathbb{H},\mathbb{R})$ with $\|g\|_\infty \leq 1$, any $t \geq 0$, and any $\mathbf{x} \in \mathbb{H}$,

$$\begin{aligned}
P_t((f - P_t(f)(\mathbf{x}))g)(\mathbf{x}) &= P_t(fg)(\mathbf{x}) - P_t(f)(\mathbf{x})P_t(g)(\mathbf{x}) \\
&= \int_0^t \partial_s P_s((P_{t-s}f)(P_{t-s}g))(\mathbf{x})\,ds \\
&= 2\int_0^t P_s(\Gamma(P_{t-s}f, P_{t-s}g))(\mathbf{x})\,ds \\
&\leq 2\int_0^t P_s(|\nabla P_{t-s}f|\,|\nabla P_{t-s}g|)(\mathbf{x})\,ds \\
&\leq 2C_1\,P_t(|\nabla f|)(\mathbf{x})\int_0^t \frac{\|g\|_\infty}{\sqrt{(t-s)}}\,ds \\
&\leq 4C_1\sqrt{t}\,P_t(|\nabla f|)(\mathbf{x}).
\end{aligned}$$

where we used the gradient bound (6) for f and the gradient bound (24) for g. The desired result follows then by $\mathrm{L}^1 - \mathrm{L}^\infty$ duality by taking the supremum over g. □

Similarly, we get also the following correlation bound for every $t \geq 0$ and $f, g \in \mathcal{C}_c^\infty(\mathbb{H},\mathbb{R})$,

$$|P_t(fg) - P_t(f)P_t(g)| \leq 2C_1^2 t\,\sqrt{P_t(|\nabla f|^2)}\sqrt{P_t(|\nabla g|^2)}. \tag{26}$$

When $f = g$, we recover the Poincaré inequality (23).

Theorem 6.3 (Yet another local Cheeger type inequality). *With the notations of* (6), *for every $t \geq 0$, every $\mathbf{x} \in \mathbb{H}$, and every ball B of $\mathbb{H}$ for the Carnot-Carathéodory metric, there exists a real constant $C_{B,t,\mathbf{x}} > 1$ such that for every function $f \in \mathcal{C}_c^\infty(\mathbb{H},\mathbb{R})$ which vanishes on B,*

$$|P_t(f)(\mathbf{x})| \leq C_{B,t,\mathbf{x}}\,P_t(|\nabla f|)(\mathbf{x}). \tag{27}$$

Proof. Let $g \in \mathcal{C}^\infty(\mathbb{H},\mathbb{R})$ be such that $\|g\|_\infty < \infty$ and $g \equiv 1$ on B^c. Since $fg = f$, the computation made in the proof of theorem 6.2 provides

$$P_t(f)(\mathbf{x}) - P_t(f)(\mathbf{x})P_t(g)(\mathbf{x}) \leq 4C_1\sqrt{t}\,\|g\|_\infty\,P_t(|\nabla f|)(\mathbf{x}).$$

For any arbitrary real number $r \geq 1$, the class of functions

$$\mathcal{C}_{B,r} = \{g \in \mathcal{C}^\infty(\mathbb{H},\mathbb{R}) \text{ with } \|g\|_\infty \leq r \text{ and } g \equiv 1 \text{ on } B^c\}.$$

is not empty since it contains the constant function $\equiv 1$. Furthermore, since $P_t(\cdot)(\mathbf{x})$ is a probability measure with non vanishing density, the following extrema

$$\alpha_-(B,r,t,\mathbf{x}) = \inf_{g\in\mathcal{C}_{B,r}} P_t(g)(\mathbf{x}) \quad\text{and}\quad \alpha_+(B,r,t,\mathbf{x}) = \sup_{g\in\mathcal{C}_{B,r}} P_t(g)(\mathbf{x})$$

are finite and non zero. Moreover, an elementary local perturbative argument on any element of the class $\mathcal{C}_{B,r}$ shows that $\alpha_-(B,r,t,\mathbf{x})\,\alpha_+(B,r,t,\mathbf{x}) < 0$ as soon as r is large enough, say $r \geq r_{B,t,\mathbf{x}}$. Thus, $P_t(f)(\mathbf{x})P_t(g)(\mathbf{x}) \leq 0$ for some $g \in \mathcal{C}_{B,r}$. The desired result follows then with $C_{B,t,\mathbf{x}} = 4C_1\sqrt{t}\,r_{B,t,\mathbf{x}}$, since one can replace f by $-f$ in the obtained inequality. Note that $C_{B,t,\mathbf{x}}$ blows up when $\mathrm{vol}(B) \searrow 0$. Actually, this proof does not use the nature of the Heisenberg group $\mathbb{H}$, and relies roughly only on the diffusion property, the smoothness of the heat kernel and the gradient bound. However, on the Heisenberg group $\mathbb{H}$, the usage of translations and dilations and of the convolution semigroup nature of $(P_t)_{t\geq 0}$ allows to precise the dependency of $C_{B,t,\mathbf{x}}$ over t and $\mathbf{x}$ by using $\mathbf{x}$-translation and $\sqrt{t}$-dilation. □

The isoperimetric content of (25) can be extracted by approximating an indicator with a smooth f, see for instance [5]. Namely, for any Borel set $A \subset \mathbb{H}$ with smooth boundary, any $t \geq 0$, and any $\mathbf{x} \in \mathbb{H}$, we get by denoting $\mu_{t,\mathbf{x}} = P_t(\cdot)(\mathbf{x})$,

$$\mu_{t,\mathbf{x}}(A)(1-\mu_{t,\mathbf{x}}(A)) \leq 2C_1\sqrt{t}\,\mu_{t,\mathbf{x}}^{\mathrm{surface}}(\partial A) \tag{28}$$

where $\mu_{t,\mathbf{x}}^{\mathrm{surface}}(\partial A)$ is the perimeter of A for $\mu_{t,\mathbf{x}}$ as defined in [1, Section 3] (see also [35]). From (27), we get similarly for any ball B in $\mathbb{H}$ and any Borel set $A \subset B^c$ with smooth boundary,

$$\mu_{t,\mathbf{x}}(A) \leq C_{B,t,\mathbf{x}}\,\mu_{t,\mathbf{x}}^{\mathrm{surface}}(\partial A). \tag{29}$$

6.3 Bobkov type isoperimetric inequalities

Let $F_\gamma : \mathbb{R} \to [0,1]$ be the cumulative probability function of the standard Gaussian distribution γ on the real line $\mathbb{R}$, given for every $t \in \mathbb{R}$ by

$$F_\gamma(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{1}{2}u^2}\,du.$$

The Gaussian isoperimetric function $\mathcal{I} : [0,1] \to [0,(2\pi)^{-1/2}]$ is defined by $\mathcal{I} = (F_\gamma)' \circ (F_\gamma)^{-1}$. The function $\mathcal{I}$ is concave, continuous on $[0,1]$, smooth on $(0,1)$, symmetric with respect to the vertical axis of equation $u = 1/2$, and satisfies to the differential equation

$$\mathcal{I}(u)\mathcal{I}''(u) = -1 \quad \text{for any } u \in [0,1] \tag{30}$$

with $\mathcal{I}(0) = \mathcal{I}(1) = 0$ and $\mathcal{I}'(0) = -\mathcal{I}'(1) = \infty$. Note that $\mathcal{I}(u) \geq u(1-u)$ for any real $u \in [0,1]$, and that $\mathcal{I}(u) \leq \min(u, 1-u))$ when u belongs to a neighborhood of $1/2$.

Lemma 6.4 (Yet another uniform gradient bound). *With the notations of* (6)*, for every* $t \geq 0$ *and* $f \in \mathcal{C}_c^\infty(\mathbb{H},(0,1))$*,*

$$\mathcal{I}(P_tf) - P_t(\mathcal{I}(f)) \leq C_1^2\sqrt{2t}\,P_t(|\nabla f|). \tag{31}$$

Proof. The inequality (31) was obtained by Bobkov in [9] for the standard Gaussian measure on $\mathbb{R}$. Later, it was generalized in [5], by using semigroup techniques, to Riemannian settings under a Γ_2 curvature assumption. We give here a proof by adapting the argument given in [5, p. 261-263] from invariant measure settings to local settings. One may assume that $\varepsilon \leq f \leq 1-\varepsilon$ for some $\varepsilon > 0$. By the diffusion property and (30)

$$\begin{aligned}
[\mathcal{I}(P_tf)]^2 - [P_t(\mathcal{I}(f))]^2 &= -\int_0^t \partial_s[P_s(\mathcal{I}(P_{t-s}f))]^2\,ds \\
&= -2\int_0^t P_s(\mathcal{I}(P_{t-s}f))P_s\left(\mathcal{I}''(P_{t-s}f)\,|\nabla P_{t-s}f|^2\right)\,ds \\
&= +2\int_0^t P_s(\mathcal{I}(P_{t-s}f))P_s\left(\frac{|\nabla P_{t-s}f|^2}{\mathcal{I}(P_{t-s}f)}\right)\,ds.
\end{aligned}$$

Next, the Cauchy-Schwarz inequality or alternatively the Jensen inequality for the bivariate convex function $(u,v) \mapsto u^2/\mathcal{I}(v) = -\mathcal{I}''(v)u^2$ gives

$$[\mathcal{I}(P_tf)]^2 - [P_t(\mathcal{I}(f))]^2 \geq 2\int_0^t [P_s(|\nabla P_{t-s}f|)]^2\,ds.$$

Now by using the gradient bound (6) we have

$$C_1\, P_s(|\nabla P_{t-s} f|) \geq |\nabla P_s(P_{t-s} f)| = |\nabla P_t f|$$

and thus

$$[\mathcal{I}(P_t f)]^2 - [P_t(\mathcal{I}(f))]^2 \geq \frac{2t}{C_1^2}\, |\nabla P_t f|^2.$$

In particular, we obtain the following uniform gradient bound

$$\|\mathcal{I}''(P_t f) |\nabla P_t f|\|_\infty = \left\| \frac{|\nabla P_t f|}{\mathcal{I}(P_t f)} \right\|_\infty \leq \frac{C_1}{\sqrt{2t}}.$$

We are now able to prove (31). By the diffusion property

$$\mathcal{I}(P_t f) - P_t(\mathcal{I}(f)) = -\int_0^t \partial_s\, P_s(\mathcal{I}(P_{t-s} f))\, ds = -\int_0^t P_s(\mathcal{I}''(P_{t-s} f)\, |\nabla P_{t-s} f|^2)\, ds.$$

By (6) we get $|\nabla P_{t-s} f|^2 \leq C_1\, |\nabla P_{t-s} f| P_{t-s}(|\nabla f|)$ and thus

$$\mathcal{I}(P_t f) - P_t(\mathcal{I}(f)) \leq C_1 \left(\int_0^t \frac{C_1}{\sqrt{2(t-s)}}\, ds \right) P_t(|\nabla f|) = C_1^2 \sqrt{2t}\, P_t(|\nabla f|).$$

□

The isoperimetric content of (31) can be extracted by approximating an indicator with a smooth f, see [5]. Namely, for any Borel set $A \subset \mathbb{H}$ with smooth boundary, any $t \geq 0$, and any $\mathbf{x} \in \mathbb{H}$, we get by denoting $\mu_{t,\mathbf{x}} = P_t(\cdot)(\mathbf{x})$,

$$\mathcal{I}(\mu_{t,\mathbf{x}}(A)) \leq C_1^2 \sqrt{2t}\, \mu_{t,\mathbf{x}}^{\text{surface}}(\partial A). \tag{32}$$

Corollary 6.5 (Yet another local Bobkov Gaussian isoperimetric inequality). *With the notations of* (6)*, for every $t \geq 0$ and $f \in \mathcal{C}_c^\infty(\mathbb{H}, (0,1))$,*

$$\mathcal{I}(P_t f) \leq P_t \left(\sqrt{(\mathcal{I}(f))^2 + 2C^4 t\, |\nabla f|^2} \right). \tag{33}$$

Proof. The desired result follows from the transportation-rearrangement argument given in [6, prop. 5 p. 427], which is inspired from [5, p. 273]. The method is not specific to the heat semigroup on the Heisenberg group. It is based in particular on a similar inequality for the standard Gaussian measure on $\mathbb{R}$ obtained by Bobkov in [10]. □

One of the most important aspect of (33) is its stability by tensor product, in contrast with (31), while maintaining the same isoperimetric content. Moreover, one may recover from (33) the Gross logarithmic Sobolev inequality (21) by using the fact that $\mathcal{I}'(u) \sim \sqrt{-2\log(u)}$ and $\mathcal{I}(u) \sim u\sqrt{-2\log(u)}$ at $u = 0$. We ignore if (33) can be obtained directly by semigroup interpolation, as for the elliptic case in [5]. The proof given in [5] for the elliptic case is based directly on a curvature bound at the level of the infinitesimal generator, which is not implied by the gradient bound (6) on $\mathbb{H}$. We ignore also if one can adapt on the Heisenberg group the two points space approach used in [10] or the martingale representation approach used in [6, 11, 24, 28]. There is a lack of a direct proof of (33) on the Heisenberg group, despite the fact that (33) and (31) are equivalent, according to the argument of Barthe and Maurey in [6, prop. 5 p. 427].

Remark 6.6 (Abstract Markov settings). *In fact, up to specific constants, most of the proofs given above have nothing to do with the group structure of the space or with the convolution semigroup nature of* $(P_t)_{t\geq 0}$. *They remain actually valid in very general settings provided that the computations make sense. The key points are a* $\sqrt{\Gamma} - P_t$ *sub-commutation and the semigroup and diffusion properties. Formally, let* L *be a diffusion operator on a smooth complete connected differential manifold* $\mathcal{M}$, *generating a Markov semigroup* $(P_t)_{t\geq 0} = (e^{tL})_{t\geq 0}$ *with smooth density with respect to some reference Borel measure on* $\mathcal{M}$. *Let* $2\Gamma f = L(f^2) - 2fLf$ *and suppose that there exists* $C : (0,\infty) \to (0,\infty)$ *such that*

$$\sqrt{\Gamma P_t f} \leq C(t)\, P_t(\sqrt{\Gamma f}) \tag{34}$$

pointwise for every $t \geq 0$ *and every smooth* $f : \mathcal{M} \to \mathbb{R}$. *Let us define* $R(t)$ *by*

$$R(t) = \int_0^t C(s) \left(\int_0^s \frac{2}{C(u)^2}\, du \right)^{-\frac{1}{2}} ds.$$

Then for every $t \geq 0$, *every* $x \in \mathcal{M}$, *and every smooth* $f : \mathcal{M} \to \mathbb{R}$,

$$P_t(|f - P_t(f)(x)|)(x) \leq 2R(t)\, P_t(\sqrt{\Gamma f})(x). \tag{35}$$

Moreover, for every $t \geq 0$ *and every smooth* $f : \mathcal{M} \to (0,1)$,

$$\mathcal{I}(P_t(f)) - P_t(\mathcal{I}(f)) \leq R(t)\, P_t(\sqrt{\Gamma f}), \tag{36}$$

and

$$\mathcal{I}(P_t(f)) \leq P_t\left(\sqrt{(\mathcal{I}(f))^2 + R(t)^2\, \Gamma f} \right), \tag{37}$$

where $\mathcal{I}$ *stands for the Gaussian isoperimetric function as in* (30). *Furthermore, if* I *is an open interval of* $\mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is a smooth convex function such that* $\varphi'' > 0$ *on* I *and* $-1/\varphi''$ *is convex on* I, *then for every* $t \geq 0$, *every* $x \in \mathcal{M}$, *and every smooth* $f : \mathcal{M} \to I$,

$$P_t(\varphi(f)) - \varphi(P_t f) \leq \left(\int_0^t C(u)^2\, du \right) P_t(\varphi''(f)\Gamma f). \tag{38}$$

Finally, if $P_t(\cdot)(x) \to \mu$ *weakly as* $t \to \infty$ *for some* $x \in \mathcal{M}$ *and some probability measure* μ *on* $\mathcal{M}$ *then the four inequalities (35-38) above hold for* μ *instead of* $P_t(\cdot)(x)$. *Here the constant in* (35) *is obtained partly by using a reverse local Poincaré inequality deduced from* (34). *On the Heisenberg group, we used an alternative constant for the reverse local Poincaré inequality, which was not deduced from* (34).

6.4 Multi-times inequalities

Let $\varphi : \mathcal{I} \to \mathbb{R}$ be fixed and as in (20). The φ-entropy functional

$$\mathbf{Ent}_\mu : f \mapsto \mathbf{Ent}_\mu(f) = \int \varphi(f)\, d\mu - \varphi\left(\int f\, d\mu \right)$$

has the tensor product property. Namely, if $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ is a probability measure on a product space $E = E_1 \times \cdots \times E_n$ then for every $f : E \to \mathcal{I}$ in the domain of $\mathbf{Ent}_\mu$,

$$\mathbf{Ent}_\mu(f) \leq \sum_{i=1}^n \int \mathbf{Ent}_{\mu_i}(f))\, d\mu$$

where the integrals in $\mathbf{Ent}_{\mu_i}(f)$ act only on the i^{th} coordinate. The details are given in [12]. Below, we use the notation $\mathbf{Ent}(U) = \mathbb{E}(\varphi(U)) - \varphi(\mathbb{E}(U))$ for any real random variable U taking its values in $\mathcal{I}$. Now, let $(X_t)_{t\geq 0}$ be the diffusion process on $\mathbb{H}$ generated by L, with $X_0 = 0$. Let also $F : \mathbb{H}^n \to \mathcal{I}$ be some fixed smooth function. Here $\mathbb{H}^n$ stands for the n-product space $\mathbb{H} \times \cdots \times \mathbb{H}$. Since $(X_t)_{t\geq 0}$ has independent stationary increments, i.e. is a Lévy process on $\mathbb{H}$ associated to a convolution semigroup, we have, for any finite increasing sequence $0 < t_1 < \cdots < t_n$ of fixed times,

$$\mathbf{Ent}(F(X_{t_1}, \ldots, X_{t_n})) = \mathbf{Ent}_{\mathcal{L}(Q_1,\ldots,Q_n)}(F \circ \pi)$$

where $\pi : \mathbb{H}^n \to \mathbb{H}^n$ is defined by

$$\pi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = (\mathbf{x}_1, \mathbf{x}_1 \bullet \mathbf{x}_2, \ldots, \mathbf{x}_1 \bullet \cdots \bullet \mathbf{x}_n)$$

for every $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{H}^n$, and where $Q_1, \ldots, Q_n$ are independent random variables on the Heisenberg group $\mathbb{H}$ with $\mathcal{L}(Q_i) = \mathcal{L}((X_{t_{i-1}})^{-1} X_{t_i}) = \mathcal{L}(X_{t_i - t_{i-1}})$ for every $i \in \{1, \ldots, n\}$, with $t_0 = 0$. The tensor product property of the entropy given above together with (20) gives

$$\mathbf{Ent}(F(X_{t_1}, \ldots, X_{t_n})) \leq C_1^2 \, \mathbb{E}_{\mathcal{L}(X_{t_1},\ldots,X_{t_n})}(\varphi''(F)\mathcal{D}^2_{t_1,\ldots,t_n} F)$$

where C is as in (6) and (20), and where

$$\mathcal{D}^2_{t_1,\ldots,t_n} F = \sum_{i=1}^{n} (t_i - t_{i-1}) \, |\nabla_i (F \circ \pi)|^2 \circ \pi^{-1}$$

where ∇_i denote the left invariant gradient ∇ of $\mathbb{H}$ acting on the i^{th} coordinate $\mathbf{x}_i$. Only the distribution of $\varphi''(F)\mathcal{D}^2_{t_1,\ldots,t_n} F$ under $\mathcal{L}(X_{t_1}, \ldots, X_{t_n})$ is of interest. Similarly, by using an argument of Bobkov detailed for instance in [6, p. 429-430], we get from (33), for any smooth function $F : \mathbb{H}^n \to (0, 1)$, by denoting $\nu = \mathcal{L}(X_{t_1}, \ldots, X_{t_n})$,

$$\mathcal{I}(\mathbb{E}_\nu(F)) \leq \mathbb{E}_\nu \left(\sqrt{(\mathcal{I}(F))^2 + 2C_2^4 \, \mathcal{D}^2_{t_1,\ldots,t_n} F} \right).$$

We ignore if such a cylindrical approach leads to functional inequalities for the paths space on $\mathbb{H}$, i.e. for the hypoelliptic Wiener measure, by letting $n \to \infty$. It sounds interesting to try to make a link with [17].

References

[1] L. Ambrosio, *Some Fine Properties of Sets of Finite Perimeter in Ahlfors Regular Metric Measure Spaces.* Advances in Mathematics 159, 51–67 (**2001**)

[2] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer, *Sur les inégalités de Sobolev logarithmiques.* Panoramas et Synthèses, 10. Société Mathématique de France, Paris, **2000**. xvi+217 pp.

[3] D. Bakry, *On Sobolev and logarithmic Sobolev inequalities for Markov semigroups.* Taniguchi symposium. New trends in stochastic analysis (Charingworth, 1994), 43–75, World Sci. Publ., River Edge, NJ, **1997**.

[4] D. Bakry, *L'hypercontractivité et son utilisation en théorie des semigroupes.* Lectures on probability theory (Saint-Flour, **1992**), 1–114, Lecture Notes in Math., 1581, Springer, Berlin, **1994**.

[5] D. Bakry and M. Ledoux, *Lévy-Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator.* Invent. Math. 123 (**1996**), no. 2, 259–281.

[6] F. Barthe and B. Maurey, *Some remarks on isoperimetry of Gaussian type.* Ann. Inst. H. Poincaré Probab. Statist. 36 (**2000**), no. 4, 419–434.

[7] F. Baudoin, *An introduction to the geometry of stochastic flows.* Imperial College Press, London, **2004**. x+140 pp.

[8] W. Beckner, *A generalized Poincaré inequality for Gaussian measures.* Proc. Amer. Math. Soc. 105 (**1989**), no. 2, 397–400.

[9] S.G. Bobkov, *A functional form of the isoperimetric inequality for the Gaussian measure.* J. Funct. Anal. (**1996**) 135 39–49.

[10] S.G. Bobkov, *An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space.* Ann. Probab. 25 (**1997**), no. 1, 206–214.

[11] M. Capitaine and E.P. Hsu and M. Ledoux, *Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces.* Electron. Comm. Probab. 2 (**1997**), 71–81.

[12] D. Chafaï, *Entropies, convexity, and functional inequalities: on Φ-entropies and Φ-Sobolev inequalities.* J. Math. Kyoto Univ. 44 (**2004**), no. 2, 325–363.

[13] D. Chafaï, *Binomial-Poisson entropic inequalities and the $M/M/\infty$ queue.* ESAIM Probab. Stat. 10 (**2006**), 317–339

[14] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian.* Problems in analysis (Papers dedicated to Salomon Bochner, **1969**), pp. 195–199. Princeton Univ. Press, Princeton, N. J., **1970**.

[15] Th. Coulhon and L. Saloff-Coste, *Isopérimétrie pour les groupes et les variétés.* Rev. Mat. Iberoamericana 9 (**1993**), no. 2, 293–314.

[16] B.K. Driver and T. Melcher, *Hypoelliptic heat kernel inequalities on the Heisenberg group.* J. Funct. Anal. 221 (**2005**), no. 2, 340–365.

[17] P. Friz, H. Oberhauser, *Isoperimetry and Rough Path Regularity*, preprint `arXiv:0711.0163v1 [math.PR]`.

[18] B. Gaveau, *Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents.* Acta Math. 139 (**1977**), no. 1-2, 95–153.

[19] R. Beals, B. Gaveau, P. Greiner, *Hamilton-Jacobi theory and the heat kernel on Heisenberg groups*, J. Math. Pures Appl. 79, 7 (**2000**) 633-689

[20] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces.* Based on the **1981** French original. With appendices by M. Katz, P. Pansu and S. Semmes. Progress in Mathematics, 152. Birkhuser Boston, Inc., Boston, MA, **1999**. xx+585 pp.

[21] L. Gross, *Logarithmic Sobolev inequalities.* Amer. J. Math. 97 (**1975**), no. 4, 1061–1083.

[22] L. Gross, *Logarithmic Sobolev inequalities and contractivity properties of semi-groups.* Varenna (**1992**). Lecture Notes in Math. 1563 (**1993**) 54–88. Springer, Berlin.

[23] Y.-Z. Hu, *A unified approach to several inequalities for Gaussian and diffusion measures*, Séminaire de Probabilités, XXXIV, Springer, Berlin, **2000**, 329–335.

[24] E.P. Hsu, *Stochastic analysis on manifolds.* Graduate Studies in Mathematics, 38. American Mathematical Society, Providence, RI, **2002**. xiv+281 pp.

[25] Juillet, N. *Geometric inequalities and generalized Ricci bounds on the Heisenberg group*, preprint (**2006**).

[26] R. Latała and K. Oleszkiewicz, *Between Sobolev and Poincaré.* Geometric aspects of functional analysis, 147–168, Lecture Notes in Math., 1745, Springer, Berlin, **2000**.

[27] M. Ledoux, *A simple analytic proof of an inequality by P. Buser.* Proc. Amer. Math. Soc. 121 (**1994**), no. 3, 951–959.

[28] M. Ledoux, *The geometry of Markov diffusion generators.* Probability theory. Ann. Fac. Sci. Toulouse Math. (6) 9 (**2000**), no. 2, 305–366.

[29] H.-Q. Li, *Estimation optimale du gradient du semi-groupe de la chaleur sur le groupe de Heisenberg.* J. Funct. Anal. 236 (**2006**), no. 2, 369–394.

[30] H.-Q. Li, *Estimations asymptotiques du noyau de la chaleur sur les groupes de Heisenberg*, CRAS Ser. I, 497-502, (**2007**).

[31] H.-Q. Li, *Estimations optimales du noyau de la chaleur sur les groupes de type Heisenberg*, preprint, personal communication, (**2007**).

[32] P. Maheux, L. Saloff-Coste, *Analyse sur les boules d'un opérateur sous-elliptique*, Math. Ann., 303, (**1995**), 713-740.

[33] T. Melcher, *Hypoelliptic heat kernel inequalities on Lie groups*, PhD thesis, University of California, San Diego (**2004**).

[34] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications.* Mathematical Surveys and Monographs, 91. American Mathematical Society, Providence, RI, **2002**. xx+259 pp.

[35] R. Monti, F. Serra Cassano, *Surface measures in Carnot-Carathéodory spaces*, Calc. Var. 13, 339 -376 (**2001**)

[36] D. Neuenschwander, *Probabilities on the Heisenberg group. Limit theorems and Brownian motion.* Lecture Notes in Mathematics, 1630. Springer-Verlag, Berlin, **1996**. viii+139 pp.

[37] M.-K. von Renesse and K.-Th. Sturm, *Transport inequalities, gradient estimates, entropy, and Ricci curvature.* Comm. Pure Appl. Math. 58 (**2005**), no. 7, 923–940.

[38] N. Th. Varopoulos, *Small time Gaussian estimates of heat diffusion kernels. I. The semigroup technique.* Bull. Sci. Math. 113 (**1989**), no. 3, 253–277.

[39] N. Th. Varopoulos and L. Saloff-Coste and Th. Coulhon, *Analysis and Geometry on Groups*, Cambridge Tracts in Math., vol. 100, Cambridge Univ. Press, Cambridge, **1992**.

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